A Systematic Approach to Deep Inference for Modal Logic

Structures and Deduction Abstract

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The formalisms of deep inference are built upon the idea that proofs and deductions should contain as little bureaucracy as possible. For example, deductions in the *calculus of structure* [3] are simply lists of formulas determined by rules which constitute a rewrite system on formulas. The formalism of *open deduction* [4] takes this one step further by only considering rewriting on a localised level, embedded in the structure of a derivation. This simple structure of derivations as linear chains makes it easy to analyse and manipulate them in a very general manner; it allows us, for example, to compose, decompose or invert derivations very easily. While decomposition theorems are very much useful for showing the admissibility of rules for a certain fragment of rules (including cut-elimination), they also allow us to disentangle certain kinds of deductions from one another. This becomes more obvious in systems beyond propositional logic, such as linear logic, where we can separate the usage of the modalities from the operational rules [5]. It is therefore interesting to consider deep inference for modal logics, which would also allow us to observe how modal reasoning in general is separable from logical operational reasoning.

Another motivation for why modal logic is an interesting thing to study for deep inference is the fact that modalities in general do not behave nicely within the proof theoretic study of using Gentzen sequents, be it due to modal rules complicating cut-elimination [2] or even the non-existence thereof [10]. However, already early on, the study of modalities in deep inference started to diverge towards the study of nested sequents [1]. Thus, most applications of deep inference to modal logic have been rather few and anything but recent. We therefore propose in this ongoing work a systematic study of classical modal logics using the open deduction formalism.

We define our logical language $\mathcal{L}$ using a countable set of propositional atoms $\texttt{Prop} = \{a, b, c, ...\}$ and writing formulas in negation normal form.

$$\mathcal{L} \ni A ::= \top \mid \bot \mid a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$$

We define negation of a formula $\bar{A}$ inductively by De Morgan dualities and double negation elimination: $\bar{\top} = \bot$, $\bar{\bot} = \top$, $\overline{A \wedge B} = \bar{A} \vee \bar{B}$, $\overline{A \vee B} = \bar{A} \wedge \bar{B}$,

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$$\mathsf{i}\!\downarrow \frac{\top}{A \vee \bar{A}} \qquad \mathsf{i}\!\uparrow \frac{A \wedge \bar{A}}{\bot} \qquad \mathsf{w}\!\downarrow \frac{\bot}{A} \qquad \mathsf{w}\!\uparrow \frac{A}{\top} \qquad \mathsf{c}\!\downarrow \frac{A \vee A}{A} \qquad \mathsf{c}\!\uparrow \frac{A}{A \wedge A}$$

$$\mathsf{s}\, \frac{(A \vee C) \wedge B}{(A \wedge B) \vee C} \qquad \mathsf{m}\, \frac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}$$

$$\mathsf{k}\!\downarrow \frac{\Box(A \vee B)}{\Box A \vee \Diamond B} \qquad \mathsf{k}\!\uparrow \frac{\Diamond A \wedge \Box B}{\Diamond(A \wedge B)} \qquad \Box\mathsf{w}\!\downarrow \frac{\bot}{\Box\bot} \qquad \Diamond\mathsf{w}\!\uparrow \frac{\Diamond\top}{\top}$$

$$\Box\!\uparrow \frac{\Box A \vee \Box B}{\Box(A \vee B)} \qquad \Diamond\!\downarrow \frac{\Diamond(A \wedge B)}{\Diamond A \wedge \Diamond B}$$

Figure 1: Rules for SKS(K)

$\overline{\Box A} = \Diamond\bar{A}$, $\overline{\Diamond A} = \Box\bar{A}$ and $\bar{\bar{a}} = a$. While it is not necessary to restrict ourselves to negation normal forms, it makes the definition of our calculus easier by needing fewer rules for connectives.

Derivations $\mathcal{D}$ which include formulas of our language $\mathcal{L}$ are defined by

$$\mathcal{D} ::= \top \mid \bot \mid a \mid \bar{a} \mid \mathcal{D} \wedge \mathcal{D} \mid \mathcal{D} \vee \mathcal{D} \mid \Box\mathcal{D} \mid \Diamond\mathcal{D} \mid \rho\, \frac{\mathcal{D}}{\mathcal{D}}$$

where $\rho\, \frac{\mathcal{D}}{\mathcal{D}}$ is a rule instance of the rule ρ. The assumption and conclusion of a derivation are defined inductively over the structure of that derivation: If $\mathcal{D} = A$ is a formula, then A is already the assumption and the conclusion. If $\mathcal{D} = \rho\, \frac{\mathcal{D}_1}{\mathcal{D}_2}$, then the assumption is given by the assumption of $\mathcal{D}_1$ with the conclusion given by the conclusion of $\mathcal{D}_2$. If either $\mathcal{D} = \mathcal{D}_1 \wedge \mathcal{D}_2$, $\mathcal{D} = \mathcal{D}_1 \vee \mathcal{D}_2$, $\mathcal{D} = \Box\mathcal{D}_1$ or $\mathcal{D} = \Diamond\mathcal{D}_1$ with the assumption of $\mathcal{D}_1$ being A_1 and the assumption of $\mathcal{D}_2$ being A_2, then the assumption of $\mathcal{D}$ is respectively $A_1 \wedge A_2$, $A_1 \vee A_2$, $\Box A_1$ or $\Diamond A_1$.

We motivate the rules of our system by essentially observing for every logical connective how it interacts with the modalities in the basic modal logic K. For example, the $\Box$ modality only distributes one way over the disjunction $\vee$ while it fully distributes over the conjunction $\wedge$. This yields the rule we call $\Box\!\uparrow$ as well as the equation $\Box(A \wedge B) = \Box A \wedge \Box B$. We call this calculus SKS(K) (for *symmetric classical system for* K), see figures 1 and 2. Note that every up-rule is dual to its down-rule, as forming the contrapositive rule of one yields and instance of the other rule.

Despite the lack of current research on this matter, there are some good first results on deep inference systems for modal logics of the classical modal cube [7, 8], hybrid logic [9] and subatomic modal logic [6]. However, considering only the systems for the basic normal modal logic K already yields somewhat different

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C \qquad A \wedge B = B \wedge A \qquad A \wedge \top = A$$

$$A \vee (B \vee C) = (A \vee B) \vee C \qquad A \vee B = B \vee A \qquad A \vee \bot = A$$

$$\Box\top = \top \qquad \top \vee \top = \top \qquad \Diamond\bot = \bot \qquad \bot \wedge \bot = \bot$$

$$\Box(A \wedge B) = \Box A \wedge \Box B \qquad \Diamond(A \vee B) = \Diamond A \vee \Diamond B$$

Figure 2: Equational theory of SKS(K)

calculi. For example, while all systems contain a K-rule of the form $\mathsf{k}\downarrow$ (see figure 1), the system SKS-K by Stewart and Stouppa [7] only contains the the further rule $\dfrac{\Box(A \wedge B)}{\Box A \wedge \Box B}$ with Morales' variation SK$^{\mathsf{K}}$S [6] using $\dfrac{\Diamond A \vee \Diamond B}{\Diamond(A \vee B)}$ instead. It may be argued that these systems, as proper subsystems of SKS(K), are therefore not actually symmetric although they are extensions of the symmetric classical system SKS for classical propositional logic.

Previous results for these calculi mainly include soundness, completeness, and cut-elimination (see [7, 8]), from which completeness of SKS(K) follows directly as it subsumes the others. However, these previously known results factor through the completeness of sequent and hypersequent calculi. Thus, our present ongoing work focusses on general results we can prove for this system.

As a first result, it is easy to show atomicity, i.e. showing that all structural rules can be reduced to structural rules on literals. Further results, which are part of ongoing research, are decomposition theorems for SKS-(K) and thus proving cut-elimination purely within our system. This also allows us to partition the modal rules into an up- and down-fragment. Both of these are shown to be equivalent to each other by showing that an up-rule $r\!\uparrow \dfrac{A}{B}$ is derivable from its down rule and vice-versa in the usual way:

$$= \dfrac{\bar{B}}{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow \dfrac{\top}{r\!\uparrow \dfrac{A}{B} \vee \bar{A}} \wedge \bar{B}}{= \dfrac{\mathsf{i}\!\uparrow \dfrac{B \wedge \bar{B}}{\bot} \vee \bar{A}}{\bar{A}}}}$$

As mentioned in the introduction, we also expect to find some kind of separation between modal and non-modal rules; however, it is unclear as of yet how far this separation goes.

Further investigations naturally include the study of extensions of the logic K and how their additional rules might interact with the system. One interesting case is the study of the logic $\mathsf{S5}$ whose deep inference presentation in [7] is not cut-free complete (a similar issue encountered in nested sequents [1, Fact 2]). The question we face is whether the additional rules and equations native to SKS-(K) can omit this problem.

Finally, let us mention for future work the possibility to extend this setup to logics which are not inside the classical modal cube. This might include other classical modal logics such as Gödel-Löb provability logic (GL), non-normal modal logics below K, intuitionistic modal logics or substructural modal logics.

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